A Practical Introduction to Commodity Option Pricing

Martin Burke (with Claude)

Draft

Abstract

A commodity is not a stock. A stock is a claim that lives in a brokerage account and, under the risk-neutral measure, grows at the funding rate; a barrel of oil is a *thing* that sits in a tank, costs money to store, pays no dividend, and is bought by refineries rather than investors. That single difference—the underlying is something you *store* and *consume*—reorganises the whole of option pricing. The spot price is often not even a tradable object, so the market prices a *forward curve* of futures instead, its shape set by storage costs, convenience yield and seasonality; prices mean-revert toward the cost of production instead of compounding; and volatility depends on *which point of the curve* you look at, rising sharply as a contract approaches delivery. This note builds commodity option pricing from that starting point, for a reader who knows Black–Scholes-level risk-neutral pricing but no commodities. We establish the cost-of-carry relation and where it breaks, show why the futures—a martingale under the pricing measure—is the natural underlying, and derive Black-76, the workhorse vanilla formula. We then put dynamics under the curve: the Schwartz one-factor mean-reverting model and its saturating variance, the two-factor Schwartz–Smith model that lets the short end and the long end move separately, and deterministic seasonality. The last third treats what the desks actually trade: Asian options on settlement averages, spread options on refining and generation margins where the correlation is the price, swing and storage contracts that are options on *volume*, and the power market, where storage is impossible, the carry argument dies, and spot dynamics turn spiky enough that the whole toolkit has to be rebuilt on forwards.

Contents

1 Why a commodity is not a stock

Every pricing formula in this note descends from a handful of physical facts about the underlying, so we start there, with the differences between a commodity and the underlyings the standard theory was built for.

1.1 Claims versus things

An equity is a *claim*: a legal abstraction that lives in an account, costs nothing to hold beyond funding, and can be sold short by borrowing it. An interest rate is not even a claim; it is a *number*, an index computed from traded instruments, and everything written on it settles in cash. A commodity is a *thing*. Crude oil fills tanks at Cushing, Oklahoma; natural gas sits compressed in salt caverns; copper stacks up in bonded warehouses; corn deteriorates in silos. Holding the underlying means renting physical space, paying insurance, and accepting that the asset may degrade. Selling it short means borrowing a physical object from someone who has it—often impossible in any size, because the people who hold inventory hold it for a reason.

That reason is the second difference: *who owns the underlying and why*. Equities are held by investors who will happily lend them out for a fee. Commodity inventories are held overwhelmingly by *consumers and processors*—refineries, utilities, smelters, millers—who hold them as an input to a production process. Their inventory is not idle capital looking for a return; it is insurance against running out. This will matter enormously when we meet the convenience yield in §2: inventory holders behave as if the physical barrel pays them a dividend that a futures contract does not, because a barrel in your own tank can keep the refinery running through a supply disruption and a futures contract cannot.

The third difference is that for many commodities *the spot price is barely a tradable object at all*. "Spot WTI" means a particular grade of crude, at a particular delivery point, changing hands for prompt delivery—a transaction that requires pipeline access, storage and a physical counterparty. An option trader cannot hedge by rapidly buying and selling barrels the way an equity trader buys and sells shares. What actually trades in size, on screens, at low cost, is the *futures contract*. The practical consequence is worth stating as a slogan we will use throughout:

> *In commodities, the hedgeable underlying is the futures, not the spot. Option pricing starts from the forward curve.*

1.2 A tour of the underlyings: families, grades, benchmarks

"Commodities" is an umbrella over markets that differ from one another almost as much as they differ from equities, and the differences that matter for pricing all trace back to physical properties. A quick tour, family by family.

Energy. Crude oil and its refined products (gasoline, diesel and heating oil, jet fuel), natural gas, coal, and electricity. Crude is storable at moderate cost and travels the world by tanker and pipeline. Natural gas is expensive to store (it must be compressed or liquefied) and expensive to move, so its markets are regional and strongly seasonal. Electricity, as we keep noting, cannot be stored at all. Energy is the largest and most actively traded family, and the one where the storage physics bites hardest.

Metals. Two very different sub-families. *Precious* metals (gold, silver) are compact and imperishable—storing a fortune in gold costs a vault and almost nothing else relative to its value—and they are held in size by investors. That makes them behave like financial assets: the carry relation of §2 holds as a near-equality and much of the equity toolkit applies. *Industrial* (base) metals—copper, aluminium, zinc, nickel—are inputs to construction and manufacturing, held in warehouses by producers and fabricators, and behave like true consumption commodities.

Agriculture. Grains and oilseeds (corn, wheat, soybeans), the "softs" (coffee, sugar, cocoa, cotton), and livestock. The defining feature is the *crop year*: supply arrives in a burst at harvest and must be stored and drawn down until the next one, so the calendar structure of §8 is built into the market's bones. Storability ranges from a year or two (grain in a silo, with quality slowly degrading) down to essentially zero (live cattle stay alive on their own schedule).

Across the families, four physical properties do most of the differentiating, and each maps to a piece of the pricing machinery built later: storage cost *relative to value* (cheap for gold, punishing for gas, infinite for power—this sets where a market sits on the spectrum of §2.4); *perishability* (grain degrades, metal does not); *transportability* (oil moves anywhere, gas and power are trapped in regional networks); and the *speed of the supply response* (a farmer responds in one season, a copper mine in a decade—this sets the strength of mean reversion, §6).

Fungibility. A market can only trade a commodity at a single price if one unit of it is as good as another, so that buyers do not care which particular barrel they receive. This property is called **fungibility**, and it has to be manufactured. Crude oil from different fields differs in density and sulphur content; wheat differs by protein level; even gold differs by purity until refined. A futures contract solves this by *specifying* the deliverable thing precisely: a grade, a quality range, and a delivery location. "WTI" is not oil in general; it is light, low-sulphur crude delivered at Cushing, Oklahoma. Anything that differs from the specification—a different grade, a different place, a different month—is, economically, a *different commodity* whose price can and does deviate from the benchmark. The deviation is called *basis*, and managing it is a large part of practical hedging (§3).

The global benchmarks. A handful of contracts serve as the reference prices for whole industries. The ones worth recognising:

Benchmark	**Commodity**	**Exchange**	**What it prices**
WTI	light sweet crude	CME/NYMEX	US oil, delivered Cushing (inland)
Brent	light sweet crude	ICE	seaborne Atlantic-basin oil
Dubai/Oman	medium sour crude	— / DME	oil flowing to Asia
Henry Hub	natural gas	CME/NYMEX	US gas (Louisiana pipeline hub)
TTF	natural gas	ICE	European gas (Dutch virtual hub)
JKM	LNG	(assessed)	spot LNG cargoes into Asia
Gold	gold	COMEX	the investment metal
Copper, aluminium	base metals	LME	industrial metal, global warehouses
Corn, wheat, soy	grains	CBOT	US and world grain trade
Coffee, sugar, cocoa	softs	ICE	tropical producers' output

Table 1: The principal global benchmark contracts. Each fixes a grade and a delivery point; everything else in its market trades at a basis to it.

Note that a single commodity can have several benchmarks—WTI, Brent and Dubai are all crude oil—because delivery *location* matters. The spread between WTI and Brent, for instance, is the price of the logistical gap between inland US storage and the seaborne market, and it has swung from near zero to over $25 as US pipelines and export rules changed.

Storage is local. It is tempting to speak of "the" storage condition of a commodity, but tanks, caverns and silos exist at particular places, and the pricing consequences are local too. The sharpest illustration is April 2020: as demand collapsed, the tank farm at Cushing—landlocked, pipeline-fed, with finite capacity—approached full, and holders of expiring WTI futures facing physical delivery with nowhere to put the oil paid others to take it: the contract settled at −$37 a barrel. Brent, priced on seaborne cargoes that can sit in tankers anywhere on the ocean ("floating storage"), fell hard but stayed positive. Same commodity, same week, opposite tails—because the storage constraint binding at one delivery point did not bind at the other.

The same logic runs through gas (Europe's winter security is its cavern inventory, and TTF winter prices track it) and grain (harvest gluts pile up where the silos are). When §5 links option skew to inventory, the inventory that matters is the one at the contract's delivery point.

1.3 Mean reversion: prices tied to a cost of production

A stock price can plausibly double and double again; nothing anchors it. A commodity price is anchored—loosely, noisily, but genuinely—by the economics of supply. If crude trades far above the marginal cost of production, drilling accelerates, supply arrives, and the price is pushed back down; if it trades below cost, production is shut in and the price recovers. The adjustment takes months to years (wells and mines are slow), so the anchor is elastic, but over horizons that matter for long-dated options it binds. Commodity spot prices *mean-revert* toward a long-run level linked to production cost, rather than growing exponentially. Section 6 makes this quantitative; the headline is that the uncertainty in a mean-reverting price *stops growing* with horizon, which flattens long-dated option values in a way Black–Scholes intuition gets badly wrong.

1.4 Seasonality, and other structure in the calendar

Nobody expects an equity to be systematically more valuable in January. Natural gas *is* systematically more valuable in January: heating demand peaks, storage draws down, and the market knows this every year. The seasonality sits in plain sight in the forward curve—January futures trade rich to July futures years ahead of time (§8)—and it also lives in volatility: a contract delivering into the winter is more uncertain than one delivering into the spring. Power adds an intraday and weekly pattern on top. This deterministic calendar structure has no equity analogue and must be carried explicitly in any model of the curve.

1.5 Volatility attached to the curve, not the asset

For a stock there is one underlying and one volatility term structure. For a commodity there is a whole curve of futures, and each contract has its own volatility, systematically higher the closer the contract is to delivery. The pattern—the *Samuelson effect*—is that news mostly concerns the near term (a refinery outage, a cold snap, an inventory surprise), and near-dated contracts absorb it fully while far-dated contracts are cushioned by the supply response already described: producers have time to react, so the long end barely moves. We quantify this in §5 and see that it is exactly what mean reversion predicts.

1.6 The comparison in one table

	Equity	**Interest rates**	**Commodities**
Underlying is…	a claim	an index	a physical thing
Spot tradable?	yes, cheaply	n/a (cash-settled)	often not, or costly
Short the spot?	yes (borrow)	n/a	usually not
Cost of holding	funding only	—	funding + storage
Benefit of holding	dividends	—	convenience yield
Long-run drift	grows (r)	mean-reverts	mean-reverts (cost anchor)
Seasonality	none	none	often strong
Hedge instrument	the stock	swaps, futures	*the futures*
Vol structure	one term structure	by tenor	per contract; Samuelson
Natural vanilla	Black–Scholes	Black / Bachelier	**Black-76**

Table 2: The idiosyncrasies in one view. Each row is a physical or institutional fact; each fact will surface as a modelling choice later in the note.

The rest of the note follows the logic of the table top to bottom: first the object that replaces the spot (the forward curve, §2–§4), then the volatility structure (§5), then dynamics (§6–§8), then the products (§9–§12).

2 Futures, forwards, and the cost of carry

2.1 The contracts

A *forward* is a private agreement to exchange the commodity for a fixed price K at maturity T; no money moves before T. A *futures contract* is the exchange-listed version: standardised (grade, location, lot size), cleared through a clearing house, and *margined*—gains and losses are settled in cash every day, so the contract's value is swept to zero each evening. We write $F(t, T)$ for the futures price quoted at time t for delivery at T. Entering a futures position costs nothing; $F(t, T)$ is not the price of an asset but the level at which the market will currently take either side of the bet.

Futures exist as a *strip* of separate contracts, one per delivery month: at any moment WTI trades for delivery next month, the month after, and so on out for a decade. The nearest-delivery contract is called the **front** or **prompt** month. Each contract eventually expires: trading stops shortly before its delivery period, and anyone still holding it must either make/take physical delivery or settle in cash, depending on the contract's terms. A hedger or investor whose exposure continues past an expiry therefore **rolls the contract**: shortly before expiry they close the position in the expiring month (sell it, if they were long) and open the same position in a later month (buy it). The exposure is maintained, but the roll is a real trade at two different prices—the expiring month and the next month rarely trade at the same level—and its cumulative cost or benefit, discussed under *roll yield* in §2, is a first-order component of any long-horizon futures position's return.

Daily margining creates a small technical difference between futures and forward prices: a long futures position receives cash when prices rise, and if prices tend to rise when interest rates are high, those cash flows can be reinvested favourably, making the futures price sit slightly above the forward. The effect is proportional to the covariance between the commodity and interest rates and is negligible for most commodities at the maturities that trade; we follow universal practice and use "futures" and "forward" interchangeably until §12 forces more care. With deterministic rates the two are exactly equal.

2.2 The cash-and-carry argument

What ties $F(t, T)$ to the spot price S_t? For a *storable* commodity, an arbitrage whose name is literal: buy the commodity in the *cash* market (the market for immediate delivery—"cash" and "spot" are synonyms here) and *carry* it—hold it, funded and stored—until the delivery date, having sold it forward at today's futures price. **Carry** is the market's word for the all-in running cost of that holding: the interest on the money tied up, plus the storage and insurance bills, minus any benefit the holding throws off. When a futures price is high enough to reimburse the holder for every one of those costs, the market is said to be at **full carry**. The argument below shows that full carry is a *ceiling*: the futures can sit at it or below it, never above.

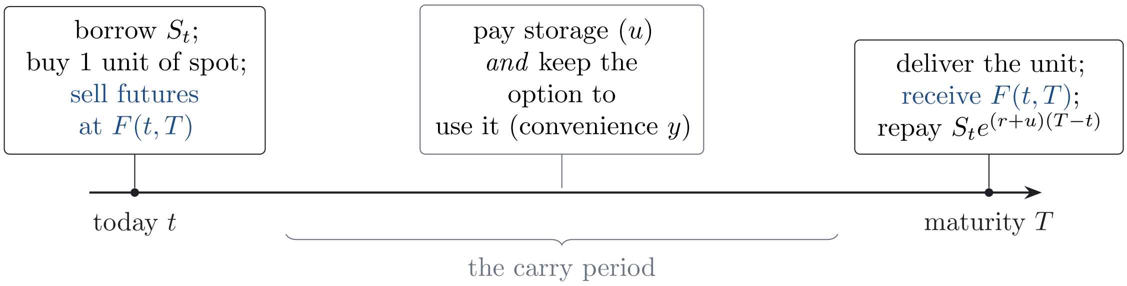

Figure 1: The cash-and-carry trade. Every leg is locked in at time t, so the terminal profit $F(t,T) - S_t e^{(r+u)(T-t)}$ is riskless: if it were positive the trade would be free money, which pins the futures price from above. The reverse trade—short the spot, invest the proceeds, buy the future—would pin it from below, but for a consumption asset the spot usually *cannot* be shorted, and the gap that opens up is the convenience yield.

Let r be the (continuously compounded) interest rate and suppose storage costs accrue at a proportional rate u per year. One bookkeeping question deserves a clear answer before writing the payoff: *who pays the storage bills between t and T, and with what money?* The arbitrageur pays them as they fall due, but not out of pocket—each bill is paid by borrowing a little more, added to the original loan of S_t. Nothing is paid at T except the final loan balance, and because interest and storage both accrue proportionally and are financed the same way, that balance compounds to $S_t e^{(r+u)(T-t)}$. (If storage is instead a fixed fee schedule, prepay it: borrow its present value U up front alongside S_t, and the debt at T is $(S_t + U)e^{r(T-t)}$; the logic is identical.) Every cash flow is thus fixed at time t—no money is needed along the way, and no unknown price ever enters.

The strategy in Figure 1 therefore costs nothing today and pays $F(t,T) - S_t e^{(r+u)(T-t)}$ with certainty at T. Ruling out riskless profit forces

$$F(t,T) \;\leq\; S_t\, e^{(r+u)(T-t)}. \tag{1}$$

For an *investment* commodity—gold is the clean example, held in size by investors who will lend it—the reverse trade also runs, and (1) becomes an equality. For a *consumption* commodity it does not: the reverse trade requires selling borrowed barrels, and the refineries holding them will not lend, because their inventory is what keeps the plant running. The futures price can therefore sit strictly *below* full carry, and for oil it usually does.

2.3 Convenience yield

The market's way of accounting for the gap is the *convenience yield* y [3]: the implicit dividend earned by holding the physical commodity rather than the paper claim on it. Writing the relation as an equality,

$$F(t,T) \;=\; S_t\, e^{(r+u-y)(T-t)}, \tag{2}$$

defines y as whatever rate makes (2) hold—exactly as dividend yield enters the equity forward, but with the sign of the economics reversed: a dividend is cash you can see, while convenience yield is the value of *optionality you hold by having the thing itself*—the ability to keep a production line running, to meet a delivery, to sell into a local squeeze. It is real but latent, and it moves: when inventories are scarce, having barrels today is precious and y is large; when tanks are brimming, one more barrel in hand is worth nothing and $y \approx 0$.

Equation (2) gives the forward curve its two canonical shapes (Figure 2):

- **Contango** ($y < r + u$): the curve rises with maturity. Typical when inventory is plentiful—holding spot has little convenience value, so futures must compensate the holder's full financing and storage bill.

- **Backwardation** ($y > r + u$): the curve falls with maturity. Typical when inventory is tight—the prompt barrel commands a premium over the deferred one because it is the one that can be used *now*.

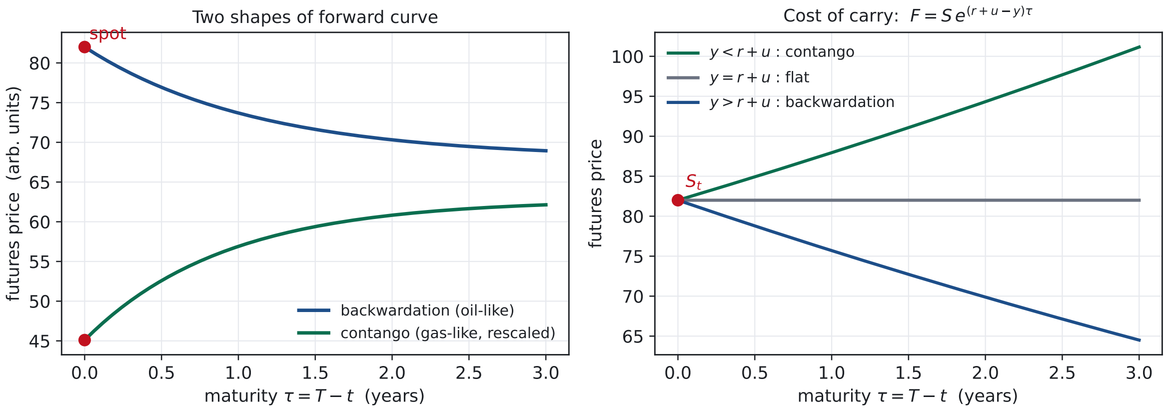

Figure 2: Left: two realistic forward-curve shapes—a backwardated, oil-like curve decaying toward its long-run level, and a contango, gas-like curve rising toward it (rescaled to share the axis). The red dot is the spot. **Right:** the building block behind them: the carry relation (2) with a *constant* net carry $r + u - y$, whose sign picks the direction. Real curves (left) are built from this block but with a net carry that varies across maturity, which is why they bend where the constant-carry fan cannot—see the text. Synthetic data.

The two panels of Figure 2 deserve a moment of reconciliation, because they can look contradictory: the right panel says carry produces exponentials that run away from the spot forever, while the left panel shows market-like curves that bend and *flatten* toward a long-run level. The resolution is that the right panel holds the net carry $r + u - y$ *constant across all maturities*—one number, one exponential—which is the correct picture over a short stretch of the curve but not over three years. In a real market each maturity carries its own effective convenience yield: the scarcity (or glut) that drives y today is expected to ease, so the near part of the curve feels a large net carry and the far part almost none, with distant prices anchored near the long-run cost of production (§6 models exactly this). A market curve is therefore best read as the constant-carry building block applied *locally*—steeply backwardated at the front where today's tightness lives, flattening out where the market expects normality to resume. The left panel is many small right-panel segments of decaying steepness chained together.

Two remarks before moving on. First, the curve's shape is a *state variable*: oil flips between contango and backwardation as global inventories swing, and §7 builds this into the dynamics by making the net carry stochastic. Second, the shape determines the cost of the *rolling* described in §2.1, independently of any view on the price level. In a backwardated market a long futures position sells each expiring contract at the higher near-month price and re-buys the cheaper next month out; repeated every month, this earns a steady gain known as *roll yield*. In contango the same routine runs in reverse—sell cheap, re-buy dear—and loses money every roll. For an investor holding commodity futures for years, this rolling gain or loss routinely rivals the price move itself in size.

2.4 Where the argument breaks

Everything above leans on one physical act: storage. Weaken it and the relation degrades gracefully; remove it and the relation dies.

- **Cheap storage** (gold): full carry holds, $y \approx 0$; the forward curve is a financing curve and the asset prices like a currency.

- **Costly, finite storage** (oil, gas, metals, grains): (2) holds with a fluctuating y; the curve carries real information about scarcity.
- **No storage** (electricity): a megawatt-hour at 6 pm today cannot be carried to tomorrow. There is no cash-and-carry, no bound (1), and *no arbitrage link at all* between today's spot power price and next January's forward. The forward becomes a risk-adjusted expectation of future spot, set by hedging supply and demand, and spot itself is free to do violent things that no storable commodity's price could sustain. Section 12 is devoted to this.

The practical summary: for storable commodities the forward curve is pinned to spot by (2) and we may treat the *curve* as the fundamental object; for power the curve is the fundamental object by default, because nothing ties it to spot.

3 Hedging in practice, and the contracts people trade

Before pricing anything, it is worth knowing concretely *who* uses these markets, *how* a hedge is actually run, and *what* the standard contracts are. This section is a field guide; every product named here is priced somewhere later in the note, and several terms used casually in the literature—calendar spreads, swing, storage deals—get their proper definitions here.

3.1 The cast of hedgers

Three archetypes generate most of the market's hedging activity.

- **Producers** (oil companies, miners, farmers) own future output and fear *falling* prices. Their natural trades: sell futures or swaps to lock a sale price, or buy put options to set a floor under revenue while keeping the upside.
- **Consumers** (airlines, utilities, food companies) must buy in the future and fear *rising* prices. Mirror image: buy futures or swaps to lock a purchase price, or buy call options to cap costs.
- **Processors** (refineries, power generators, crushers, smelters) buy one commodity and sell another, so their exposure is to the *difference* between two prices—a margin, not a level. Their hedges involve both legs at once, which is where the spread products of §10 come from.

Facing them stand speculators and index investors, who take the other side for expected return, and dealers, who intermediate and warehouse the mismatches.

3.2 Running a futures hedge: the practical frictions

On paper a producer's hedge is one line: sell futures against future output. In practice three frictions dominate the desk's life.

Margin, and the cash-flow mismatch. Recall from §2.1 that futures settle their gains and losses in cash *daily*. The physical side of the hedge does not: the oil in the ground pays nothing until it is produced and sold. So a producer short futures against next year's output faces a genuine liquidity problem if prices *rise* sharply: the futures leg demands cash margin now, while the offsetting gain—selling the physical oil at the new higher price—arrives months later. The hedge is sound in total but can bankrupt its owner on timing. This is not hypothetical: in 1993 the German firm Metallgesellschaft, running exactly this structure at enormous scale, was undone by margin calls on the futures leg and unwound near the worst point, losing over a billion dollars [12].

Basis risk. A hedge is exact only if the contract's deliverable matches the exposure—same grade, same place, same dates. Usually it does not: an airline burning jet fuel finds no liquid jet-fuel future and hedges with heating oil (a chemically similar refined product) instead; a regional gas utility hedges with Henry Hub futures though its gas is priced at a local hub. What remains after such a hedge is exposure to the *difference* between the two prices—the **basis** of §1.2. Basis risk is smaller than the outright price risk, but it is precisely the risk that cannot be removed with the liquid benchmark, and it is why the basis swaps below exist.

Maturity mismatch, and stack-and-roll. Liquidity in futures is concentrated in the first year of contracts; a producer wanting to hedge five years of output cannot simply sell five years of futures in size. The standard workaround is the **stack-and-roll**: sell the whole hedge volume "stacked" into the liquid near contracts, and roll the stack forward month after month (rolling as defined in §2.1), maintaining the total short until the physical sales arrive. This converts an unhedgeable maturity mismatch into two known exposures: the roll cost of §2 (paid every month if the curve is in contango) and the risk that near and far prices move differently—which is exactly the calendar-spread risk defined below, and one reason the two-factor models of §7 matter to hedgers, not just to option traders. Metallgesellschaft's structure was a stack-and-roll; the roll costs in a contango market compounded its margin problem.

3.3 The over-the-counter contracts

Exchange-listed futures and options are standardised. Most commodity hedging, however, is done *over the counter* (OTC): bilaterally negotiated contracts between a company and a dealer, documented under master agreements and settled in cash against published price indices. OTC contracts can match the hedger's exact volumes, dates and location—eliminating basis risk by construction—at the cost of the dealer's price margin and of managing exposure to the dealer as a counterparty. The standard instruments:

The fixed-for-floating swap. The workhorse. The two parties agree a volume, a schedule, a fixed price K, and a floating index; each period (usually a calendar month) one pays fixed, the other pays the *average of the index's daily published prices* over that period, and only the difference changes hands. Concretely: an airline swaps 10,000 barrels a month at $K = \$95$ against the monthly average of a published jet-fuel price. In a month when the average is \$102 the airline *receives* $(102 - 95) \times 10{,}000 = \$70{,}000$, offsetting the higher price it paid for physical fuel; when the average is \$88 it pays \$70,000 away, giving back what it saved at the pump. Net effect: fuel is locked at \$95 regardless, with no physical delivery and no options premium. A swap is economically a strip of forwards, one per period, settling on an *average*—a detail that matters enormously for options, as §9 explains.

Average-price options. The option form of the same idea: a call paying $(\bar{A} - K)^+$ per unit, where $\bar{A}$ is the period's average index price—a cap on cost that still lets the hedger benefit when prices fall (the producer's put floor is the mirror image). These are the Asian options of §9, and in commodities they, not options on a single date's price, are the natural vanilla. A producer wanting a floor without paying premium sells a call against the put—a **collar**, bounding the realised price within a band at zero upfront cost.

Swaptions. An option to enter a swap at a pre-agreed fixed price at a future date—used by hedgers who want to lock optionality on next year's hedging level itself. Less common than the instruments above; priced with the same Black-76 machinery applied to the forward swap price.

Basis swaps. A swap of one floating index against another (plus a fixed spread): local hub gas against Henry Hub, jet fuel against heating oil, one crude grade against another. This is the instrument that offloads the basis risk left over from a benchmark hedge (§3.2).

Calendar spreads, defined. A **calendar spread** is a position in two delivery months of the *same* commodity—long March, short September, say—whose value is the difference $F(t,T_1) - F(t,T_2)$ between the two futures prices, i.e. a bet on the *shape* of the forward curve rather than its level. A *calendar spread option* (listed on the major energy exchanges, and traded OTC) is an option on that difference. Anyone who owns storage is naturally long this spread structure—buy the cheap month, store, deliver into the dear month—which is why calendar spreads are the pricing language of storage (§11) and a core object in this note from §10 on.

Physical contracts with optionality inside. Finally, much commodity "optionality" is not written as an option at all but embedded in supply agreements: a **take-or-pay** clause obliges a buyer to pay for a minimum volume whether or not it is taken (a forward with a penalty structure); a **swing** contract lets the buyer vary the volume delivered day by day within agreed bounds at a fixed price (an option on *volume*); a **storage lease** rents cavern or tank capacity (an option on calendar spreads); and a **tolling agreement** rents a power plant—the right to turn fuel into electricity for a fee—which is an option on the spread between power and fuel prices. Sections 11–12 price all of these; the point to carry forward is that they are options, whether or not the word appears in the contract.

Instrument	**Typical user**	**What it locks / buys**
Futures	everyone	price of one delivery month
Fixed-for-floating swap	producer / consumer	average price over each period
Average-price option	producer / consumer	floor or cap on that average
Collar	producer / consumer	band, at zero premium
Swaption	sophisticated hedger	option on a future hedge level
Basis swap	anyone off-benchmark	the local-vs-benchmark gap
Calendar spread (option)	storage owners, traders	the curve's shape
Swing / take-or-pay	gas & power buyers	volume flexibility
Storage lease / tolling	asset players	the physical option itself

Table 3: The standard contracts, their users, and the risk each addresses.

4 Black-76: the vanilla workhorse

4.1 The futures price is a martingale

Here is the observation that makes commodity *vanillas*—market shorthand for the simplest standard options, European calls and puts—simpler than their equity counterparts, not harder. Under the risk-neutral measure $\mathbb{Q}$ (money-market numeraire), the price of any *traded asset* grows at r. A futures position, however, is not an asset—it is a costless bet, resettled daily. Consider holding one contract over $[t, t+\mathrm{d}t]$: it costs nothing to enter, and pays the price change $\mathrm{d}F$. A costless position must have zero expected gain under $\mathbb{Q}$, so

$$\mathbb{E}^{\mathbb{Q}}[\mathrm{d}F(t,T) \,|\, \mathcal{F}_t] = 0 \qquad \Longrightarrow \qquad F(t,T) \text{ is a } \mathbb{Q}\text{-martingale.} \tag{3}$$

The futures price has *no drift* under the pricing measure. All the messy carry terms of §2—r, u, the unobservable y—live in the *relation between spot and futures*, and vanish from the dynamics of the futures itself. By quoting you a futures price, the market has already done the carry accounting for you. This is why pricing options *on the futures* is the natural move: the one input you would otherwise have to model (the drift, via the convenience yield) is handed to you by the market.

A useful corollary, with deterministic rates:

$$F(t,T) \;=\; \mathbb{E}^{\mathbb{Q}}[S_T \,|\, \mathcal{F}_t]\,, \tag{4}$$

the futures price is the risk-neutral expectation of the spot at delivery (at $t = T$ the future and the spot converge, and a driftless process equals the expectation of its terminal value). Note carefully: risk-neutral, not real-world. The forward curve is *not* a forecast; it differs from the market's true expectation by a risk premium, exactly as an equity's forward $S_t e^{r\tau}$ is not a forecast of the stock. Whether that premium is positive or negative—whether hedging pressure comes mostly from producers selling forward (pushing futures below expected spot, "normal backwardation") or consumers buying forward—is a classic and unresolved empirical question that pricing, mercifully, never needs answered.

4.2 The formula

Take the standard European call: the right, at time T, to a payoff $(F(T,T) - K)^+ = (S_T - K)^+$. Model the futures under $\mathbb{Q}$ as driftless lognormal,

$$\frac{\mathrm{d}F(t,T)}{F(t,T)} \;=\; \sigma\,\mathrm{d}W_t, \tag{5}$$

so that $\ln F(T,T)$ is Gaussian with mean $\ln F(t,T) - \frac{1}{2}\sigma^2\tau$ and variance $\sigma^2\tau$, $\tau = T - t$. The option value is the discounted expectation of the payoff, and the same two-Gaussian-integrals computation that yields Black–Scholes gives **Black-76** [1]:

$$\begin{aligned} C &= e^{-r\tau}[\,F\,N(d_1) - K\,N(d_2)\,]\,, \qquad P = e^{-r\tau}[\,K\,N(-d_2) - F\,N(-d_1)\,]\,, \\ d_{1,2} &= \frac{\ln(F/K) \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}, \end{aligned} \tag{6}$$

with $F = F(t,T)$ and N the standard normal distribution function. Appendix B derives this properly by the martingale method, with the Black–Scholes derivation run side by side so the reader can see exactly where the two part company (they part in one place only: the mean of the lognormal).

Be clear about what the input F *is*: it is a point read off the forward curve of §2—the market's quoted price today for the delivery month the option settles on. There is no separate "underlying price" to estimate and no carry adjustment to compute; the curve *is* the underlying. An option expiring in June is priced off the June point of the curve, a December option off the December point, and a book of options across many months has the whole forward curve as its underlying—which is why everything after §5 is about modelling the curve's joint behaviour rather than any single price.

Structurally this is Black–Scholes with two changes, both simplifications. The underlying's forward value is F itself—no $e^{(r-q)\tau}$ growth factor, by the martingale property—and the discount factor $e^{-r\tau}$ sits *outside*, applied to the whole expected payoff, because the option premium is paid today while the payoff arrives at T. Interest rates thus touch the price only through that final discounting; they no longer shape the distribution of the underlying. For short-dated options on volatile commodities, rates are almost a spectator.

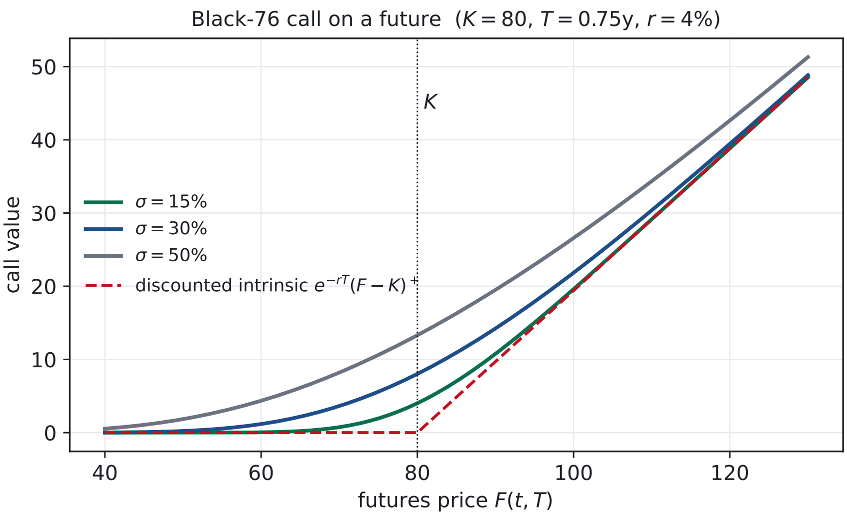

Figure 3: Black-76 call value (6) against the futures level for three volatilities, with the discounted intrinsic value in red. The familiar Black–Scholes geometry survives intact; only the carry accounting has moved out of the picture, into the quoted F.

4.3 Reading the formula on a desk

Hedging. The delta $\partial C/\partial F = e^{-r\tau}N(d_1)$ is a position in the *futures*—costless to hold, liquid, and exactly the hedgeable object identified in §1. The full Greeks are collected in Appendix B.

What σ means. It is the volatility *of the futures contract underlying the option*, not of the spot. Different maturities have genuinely different σ—not as a small correction but as first-order structure (§5).

Market conventions. Listed options on futures (CME's WTI and Henry Hub options, for instance) are typically *American*, and on some exchanges the premium is margined rather than paid up front (for a fully margined option the $e^{-r\tau}$ in (6) drops out). The American feature on a driftless underlying is worth little; the European Black-76 price is the universal quoting and intuition standard, and implied vols are quoted through it.

Which measure? A careful reader may worry that with stochastic rates, "driftless under $\mathbb{Q}$" should really be "driftless under the T-forward measure" for the forward price. True—and immaterial here: for commodities one prices with deterministic rates in almost all cases, the volatility dwarfs any rate–commodity covariance, and (6) stands. The exception, again, is long-dated structures, where the futures-vs-forward distinction and rate correlation can be revisited.

5 The volatility surface

Black-76 needs one number, σ. The market, in effect, supplies a whole table of them—a different σ for every option maturity T and every strike K—known as the volatility *surface* $\sigma(K, T)$. Two features of that surface are distinctively commodity-shaped: how volatility varies with *maturity* (§5.1) and how it varies with *strike* (§5.2).

5.1 The Samuelson effect

Fix a contract and watch its volatility as calendar time advances toward its delivery date: it *rises*, and steeply—often by a factor of two or three over the final year. This is the Samuelson effect [2], and it is worth building carefully, because it is the single most important structural fact about commodity volatility. Two ingredients produce it.

Ingredient one: news is mostly about the present. What moves commodity prices day to day is information about current physical conditions—an inventory report, a cold snap, a refinery outage, a mine strike. Each such item changes how scarce the commodity is *now* and over the weeks it takes the disruption to pass.

Ingredient two: the market expects shocks to heal. Prices for delivery far in the future are anchored by the supply response of §1: give producers two years and they can drill, plant, or restart enough capacity to offset almost any of today's surprises. So when today's news arrives, the market marks up the near contracts fully—no help can arrive by February—but marks up the two-year contract only slightly, because by then the shock is expected to have been absorbed.

Put together: *every piece of news hits the front of the curve hardest and dies away along the maturity axis.* Figure 4 (left) shows one such shock's imprint on the whole curve; since this happens with every shock, every day, a contract's day-to-day variability depends on where it sits relative to delivery. A contract three years from delivery lives at the quiet far end of every shock; the same contract, two years later, sits at the violent front and absorbs every headline in full. Its volatility has not changed because the market changed—it changed because the contract *moved along the curve* toward delivery. That is the Samuelson effect, and Figure 4 (right) shows it in the raw: two contracts driven by identical shocks, one near and one far, with utterly different variability.

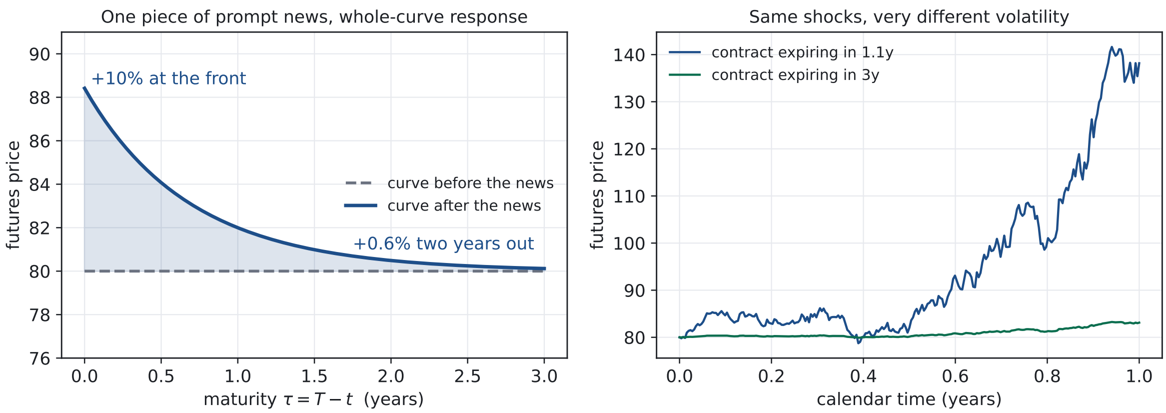

Figure 4: The mechanism behind the Samuelson effect. **Left:** a single piece of prompt news (here bullish by 10% at the front) marks up the whole forward curve, but its imprint decays along the maturity axis—the two-year point barely moves, because the market expects supply to have adjusted by then. **Right:** one year of simulated daily prices for two contracts driven by the *same* sequence of shocks, one expiring shortly after the sample ends and one far away: proximity to delivery, not any difference in the news, makes the near contract wild and the far one placid.

A convenient and, as we will see, theoretically motivated parameterisation of this decay is exponential:

$$\sigma_F(t,T) \;=\; \sigma\, e^{-\kappa(T-t)}, \tag{7}$$

where $\sigma_F(t,T)$ is the instantaneous volatility of $F(t,T)$: equal to σ at delivery, damped by $e^{-\kappa\tau}$ at maturity τ.

The Black-76 implied volatility for an option expiring at T is the root-mean-square of (7) over the option's life,

$$\begin{aligned} \bar{\sigma}(T)^2 &= \frac{1}{T}\int_0^T \sigma^2 e^{-2\kappa(T-t)}\,\mathrm{d}t \\ &= \sigma^2\,\frac{1-e^{-2\kappa T}}{2\kappa T}, \end{aligned} \tag{8}$$

a *declining* term structure: short-dated options price near the full spot volatility, long-dated options much below it. Both views—per-contract vol rising into expiry, and implied vol falling with maturity—are the same fact, shown side by side in Figure 5.

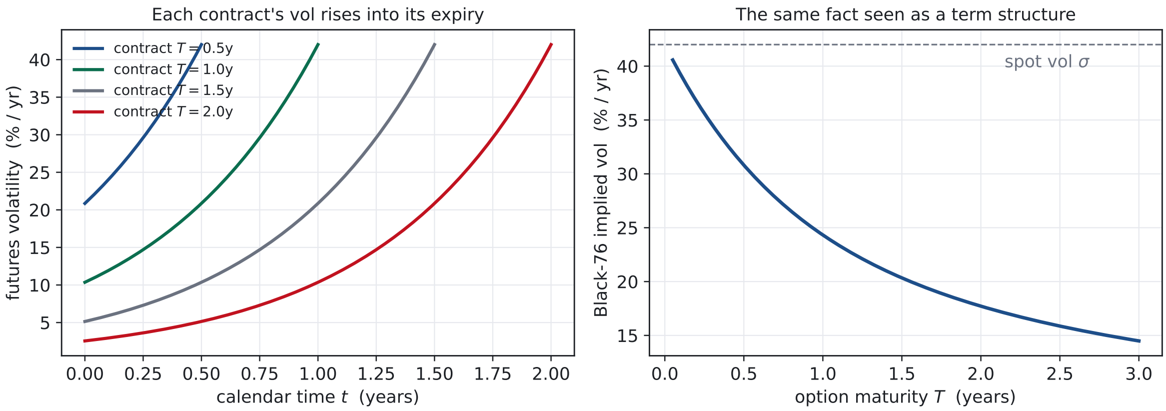

Figure 5: The Samuelson effect with $\sigma = 42\%$, $\kappa = 1.4$. **Left:** each line tracks *one* futures contract through calendar time (the horizontal axis), plotting its instantaneous volatility (7); each line ends at its own delivery date. Far from delivery a contract is quiet (the two-year contract starts the sample at about 5%/yr); as calendar time carries it toward its delivery date it slides up the same ramp every other contract rides, reaching the full short-horizon volatility $\sigma = 42\%$ just as it expires. The four lines are horizontal translates of one another: the ramp is a property of *time to delivery*, not of the calendar. **Right:** the consequence for options: the Black-76 implied volatility (8), an average of the left panel's ramp over the option's life, declines with option maturity.

One practical consequence deserves emphasis. A portfolio of commodity options is exposed not just to the overall *level* of volatility but to *where each underlying contract sits on its Samuelson ramp.* Rolling a hedge from one contract to the next (§2.1) changes the volatility of the thing held even if nothing happens in the market, simply because the new contract sits at a different point of the ramp. Commodity risk systems accordingly track volatility exposure contract by contract rather than as one number per commodity.

5.2 Smile and skew: read the inventory

First, the vocabulary, in plain terms. Take all the options trading on one futures contract, one strike at a time, and for each strike find the volatility σ that makes Black-76 reproduce that option's market price. This reverse-engineered number is the **implied volatility** of that strike. If the model's assumptions held exactly—one lognormal futures with one volatility—the procedure would return the *same* σ at every strike: a flat line. In every real market it does not. Plotted against strike, implied volatility traces a curve, called the **smile**; when the curve is lopsided—one side higher than the other—the asymmetry is called **skew**. A smile is the market saying, in the model's own units, that it considers large moves more likely (and insurance against them more valuable) than the lognormal distribution admits, and the *direction* of the tilt says which side it fears: if options struck far *below* the current price carry the highest implied volatilities, the market is paying up for protection against a collapse; if the high-strike options are dearest, it fears a spike.

In equity indices the tilt is almost always the same way: strikes below the market trade at markedly higher implied volatility, because the move everyone insures against is the crash. Commodities are more interesting: the feared side *depends on the state of inventory*, and the smile tilts accordingly (Figure 6).

- **Tight inventory: the upside is feared.** When stocks are low, small extra demand meets a market that physically cannot supply more at short notice, and the price can jump violently upward—while the downside is cushioned (some supply can always be withheld). High-strike options are the insurance against this, so their implied volatilities are pushed above the at-the-money level: the smile tilts up to the right (Figure 6, left). Natural gas heading into winter is the textbook case; the power-market spikes of §12 are the extreme version.

- **Glut: the downside is feared.** When storage is close to full, the danger runs the other way: production keeps arriving, there is nowhere to put it, and holders may have to pay to be rid of it—the April 2020 episode of §1.2, where the WTI front month settled at −\$37. Low-strike options are the insurance, and the smile tilts up to the left (Figure 6, middle).

- **Hedging flow shapes both wings.** On top of the physics sits persistent, one-directional demand: producers buy low-strike puts year after year (floors under their revenue, §3), consumers buy high-strike calls (caps on their costs). Dealers who sell both accumulate risk they cannot fully lay off and charge for it, which props up implied volatility at *both* ends relative to the middle (Figure 6, right).

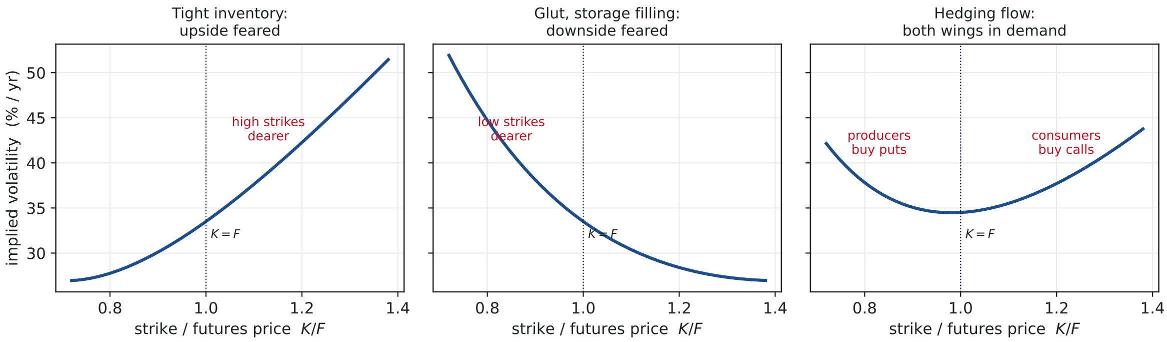

Figure 6: The three forces behind commodity smiles, sketched as implied volatility against the ratio of strike to futures price (so $K/F = 1$ is at-the-money, the dotted line). **Left:** with inventory tight, upward spikes are the feared move and high strikes carry the premium. **Middle:** with storage near full, collapse is feared and low strikes carry it. **Right:** routine hedging flow—producers buying puts, consumers buying calls—supports both wings even in calm conditions. A real surface at a given date is a blend of the three, with the tilt migrating as inventory swings.

The tilt therefore *moves* with the inventory cycle, flipping sign between scarce and glutted regimes, and traders read it as a real-time gauge of which tail the market fears—just as equity traders read their put skew as a gauge of crash fear. Modelling a full smile properly needs the local/stochastic volatility machinery familiar from equities, which we will not rebuild here; for this note's purposes the surface is an input, and the structural features to respect are the Samuelson term structure and the inventory-driven, sign-flipping tilt just described.

6 Mean reversion: the Schwartz one-factor model

Black-76 prices each contract's option in isolation, taking that contract's volatility as given. To price anything that couples maturities—calendar spreads, swing, storage, or simply a consistent surface—we need dynamics for the curve, and the first ingredient is the mean reversion promised in §1.

6.1 The model

The classic specification (Schwartz's Model 1 [6]) puts an Ornstein–Uhlenbeck process on the *log* spot price $X_t = \ln S_t$: under the pricing measure,

$$\mathrm{d}X_t \;=\; \kappa\,(\hat{\theta} - X_t)\,\mathrm{d}t \;+\; \sigma\,\mathrm{d}W_t. \tag{9}$$

The log price is pulled toward a long-run level $\hat{\theta}$ with strength κ: at distance δ from the anchor, the price closes that gap at expected rate $\kappa\delta$ per year, giving a half-life of $\ln 2/\kappa$ ($\kappa = 1.2$ means shocks halve in about seven months—a representative oil-market magnitude). The hat on $\hat{\theta}$ flags that this is the *risk-neutral* anchor: it absorbs the market price of risk, sitting below the real-world long-run level if investors demand a premium for holding commodity risk. In practice one never estimates $\hat{\theta}$ from history; it is read off the forward curve, which (4) makes the model-independent carrier of exactly this information. Calibrate the drift to the curve you can trade, not to an econometric opinion.

6.2 What mean reversion does to uncertainty

Solving (9) (Appendix C) gives a Gaussian X_T with

$$\mathrm{Var}(X_T \,|\, \mathcal{F}_t) \;=\; \sigma^2\,\frac{1 - e^{-2\kappa\tau}}{2\kappa} \quad \xrightarrow[\tau\to\infty]{} \quad \frac{\sigma^2}{2\kappa}. \tag{10}$$

For small τ this is $\sigma^2\tau$—locally the process is just as volatile as a GBM—but the variance *saturates* at $\sigma^2/2\kappa$ instead of growing without bound. Figure 7 shows both faces: paths that keep being hauled back to the anchor, and an uncertainty curve that flattens after a few half-lives.

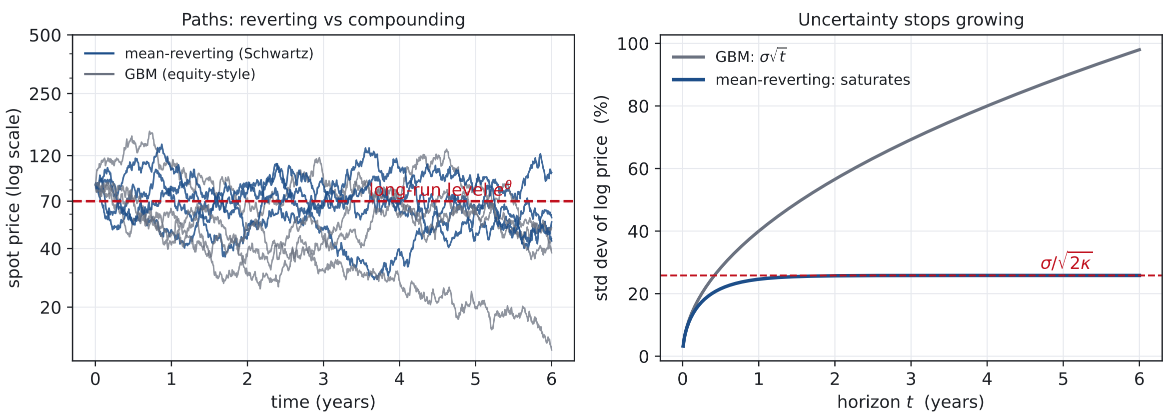

Figure 7: Mean reversion versus compounding, $\sigma = 40\%$, $\kappa = 1.2$, anchor at 70. **Left:** simulated mean-reverting (blue) and GBM (grey) spot paths on a log scale; the GBM paths disperse indefinitely, the Schwartz paths orbit the anchor. **Right:** standard deviation of the log price versus horizon: $\sigma\sqrt{t}$ for GBM against the saturating (10), with asymptote $\sigma/\sqrt{2\kappa}$ (dashed red).

The option-pricing consequence is immediate. The futures in this model is again lognormal, and its instantaneous volatility works out (Appendix C) to precisely the Samuelson form (7), $\sigma e^{-\kappa(T-t)}$; a T-maturity option's Black-76 implied vol is precisely (8). The declining term structure of Figure 5 is not a stylised fact bolted on—it is what mean reversion *is*, seen through options. A desk quoting long-dated commodity options off flat vol (the equity habit) will systematically overprice them, and by a lot: with $\sigma = 40\%$ and $\kappa = 1.2$, the five-year implied vol in (8) is about 16%, not 40%.

6.3 What one factor cannot do

Model (9) has a single shock, so it moves the whole curve in lockstep: every futures price is a deterministic function of today's spot (Appendix C), correlations between any two maturities are exactly one, and the long end of the curve is *frozen*—as τ grows, $F(t,T) \to e^{\hat{\theta}+\sigma^2/4\kappa}$, a constant. Real curves do not behave this way: the long end drifts as views on production cost evolve, the curve's slope flips between contango and backwardation, and calendar-spread options would be worthless in this model when they demonstrably are not. Fixing this needs a second factor.

7 Two-factor models: letting the curve breathe

7.1 Schwartz–Smith: short-term deviation, long-term level

The cleanest specification is Schwartz–Smith [7]. Split the log spot into two unobserved parts,

$$\ln S_t \;=\; \chi_t + \xi_t, \qquad \begin{aligned} d\chi_t &= -\kappa\,\chi_t\,dt + \sigma_\chi\,dW_t^\chi \quad &&\text{(short-term deviation)},\\ d\xi_t &= \mu\,dt + \sigma_\xi\,dW_t^\xi &&\text{(long-term level)}, \end{aligned} \tag{11}$$

with correlation $\rho\,dt$ between the Brownian motions. The reading is physical. ξ is the slow-moving equilibrium price—the market's evolving view of long-run production cost—and follows a random walk: permanent shocks (technology, resource depletion, OPEC regime change) live here. χ is the transient part—inventory surprises, weather, outages—and decays at rate κ: temporary shocks live here and are expected to wash out. Mean reversion in this model is not reversion to a fixed number but to a *moving* anchor ξ_t, which is both more realistic and more honest about how little we know about the long run.

Because both factors are Gaussian, the futures curve is closed-form (Appendix C):

$$\ln F(t,T) \;=\; e^{-\kappa\tau}\,\chi_t \;+\; \xi_t \;+\; A(\tau), \qquad \tau = T - t, \tag{12}$$

with $A(\tau)$ a deterministic function absorbing drifts and variances. The structure of (12) is the payoff of the whole construction: the log curve loads on the short factor with weight $e^{-\kappa\tau}$ and on the long factor with weight 1. A shock to χ whips the front of the curve and leaves the back almost still; a shock to ξ shifts the whole curve in parallel (Figure 8). These are precisely the first two principal components—level and slope-at-the-front—that dominate empirical futures-curve moves, which is why two factors, not one and not five, became the industry default.

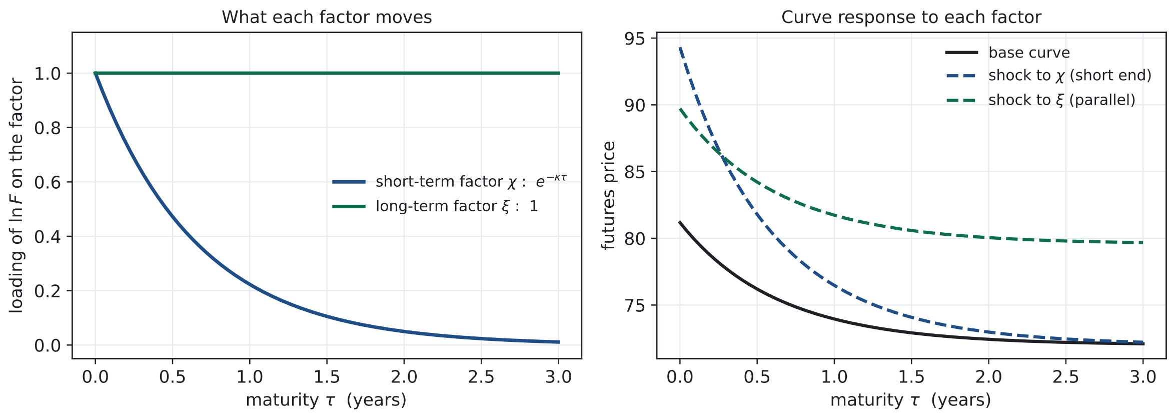

Figure 8: The Schwartz–Smith curve (12) with $\kappa = 1.5$. **Left:** factor loadings of $\ln F$ across maturity: $e^{-\kappa\tau}$ for the short-term deviation χ, constant 1 for the equilibrium level ξ. **Right:** a base backwardated curve and its response to a positive shock in each factor—χ moves the short end and decays across maturity, ξ moves the curve in parallel.

The futures volatility now has a floor. From (12),

$$\sigma_F(\tau)^2 \;=\; \sigma_\chi^2 e^{-2\kappa\tau} + \sigma_\xi^2 + 2\rho\,\sigma_\chi\sigma_\xi\,e^{-\kappa\tau} \;\xrightarrow[\tau\to\infty]{}\; \sigma_\xi^2. \tag{13}$$

The Samuelson ramp survives at the front (the $e^{-2\kappa\tau}$ term), but the long end keeps a genuine volatility σ_ξ—the long-run anchor itself is uncertain—matching what long-dated options actually cost. And because the two ends of the curve are no longer perfectly correlated, calendar-spread volatility is positive: the products of §10 become priceable.

7.2 Gibson–Schwartz, and how to think about the zoo

The older Gibson–Schwartz model [4] reaches the same place by a different door: keep the spot lognormal, but make the *convenience yield* itself an Ornstein–Uhlenbeck process, entering the spot drift as $r - \delta_t$. A stochastic convenience yield is a stochastic slope for the forward curve, so contango/backwardation flips become a state variable, as promised in §2. Schwartz and Smith showed the two models are equivalent—a linear change of variables maps one onto the other—so the choice is one of interpretation: Gibson–Schwartz speaks the language of carry (S and δ), Schwartz–Smith the language of shocks (transient and permanent). Gabillon's model [5], formulated directly on front and long prices, is a third seat in the same room. For pricing, what matters is invariant across them: two Gaussian factors, exponential loadings, closed-form futures, Black-76-style option formulas with maturity-dependent vol (13).

Calibration in practice runs in two sweeps: fit today's curve exactly (the factors' current values, or equivalently a deterministic shift, absorb it—the same trick as fitting a rates model to the discount curve), then fit $(\kappa, \sigma_\chi, \sigma_\xi, \rho)$ to the option market: the term structure of at-the-money implied volatility (strike equal to the futures price) pins the vol parameters, and calendar-spread option prices pin the correlation structure. When no options trade, a Kalman filter over the history of curve moves does the job—the state-space machinery of the companion Kalman note applies verbatim, with (12) as the observation equation.

8 Seasonality

For gas, power and agriculture, the calendar is not a nuisance but a headline feature, and it enters the model exactly where the data says it lives: in the *deterministic* part of the curve. Write

$$\ln F(t,T) \;=\; s(T) \;+\; e^{-\kappa\tau}\chi_t \;+\; \xi_t \;+\; A(\tau), \tag{14}$$

where $s(\cdot)$ is a periodic function of the *delivery date* T—not of t, and not of τ. That placement is the entire content of the modelling decision, and it is worth dwelling on. A January contract is expensive because *January* is expensive; its premium does not decay as the contract ages, and it does not attach to whichever contract happens to be prompt. Seasonality rides through the curve with the delivery month.

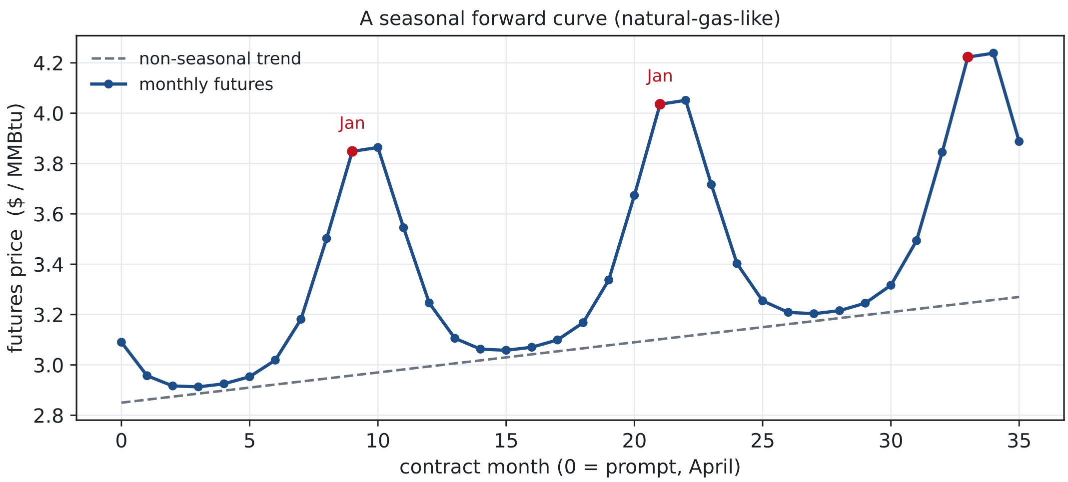

Figure 9: A natural-gas-like forward curve: 36 monthly contracts on a gentle contango trend (grey dashed), with winter delivery months commanding a premium peaking in January (red dots). The seasonal shape is knowable years ahead and sits in the deterministic $s(T)$ of (14); the stochastic factors move the curve around it. Synthetic data.

Three working rules follow.

Never smooth through the humps. A curve-fitting routine that spline-smooths the winter premium away will manufacture arbitrages against every calendar spread on the exchange. The seasonal shape is signal.

Volatility is seasonal too. A February gas contract is more volatile than a May contract *at the same time to expiry*: winter demand is weather-driven and inelastic, so the same supply shock moves winter prices further. In the models, σ_χ (or the Samuelson σ) acquires a periodic dependence on delivery month. Option desks quote gas vol month by month for exactly this reason.

Seasonality and mean reversion are different clocks. Mean reversion damps shocks in *time since the shock*; seasonality is periodic in *delivery date*. Conflating them—e.g. estimating κ on data with the seasonal cycle left in—produces spurious ultra-fast reversion as the estimator strains to explain the annual cycle. Deseasonalise first, then estimate dynamics.

9 Asian options: the commodity vanilla

We now turn to pricing the products of §3.3, and begin with the one that is quietly the most traded commodity option in the world. In equities an option on an *average* price (an "Asian" option, in the market's unhelpful jargon) is a niche exotic; in commodities the average is the natural settlement object—the swaps of §3.3 settle on monthly averages, and the average-price options written alongside them make up the bulk of producer and consumer option hedging.

9.1 Why averages

The reasons are physical and institutional at once. A refinery buys crude every week of the month, so its cost is the monthly *average* price and the matching hedge is an option on that average. Physical contracts and price benchmarks are themselves defined as averages over pricing windows (a month of settlement prices), partly to be representative of the period's trade and partly because an average over many days is far harder for anyone to manipulate than a single day's closing price. So the averaging is not an exotic flourish; it is the shape of the exposure.

9.2 Averaging tames the distribution

The payoff is $(A_T - K)^+$ with A_T the arithmetic average of the settlement prices over a window. The first-order fact is that A_T is much less volatile than S_T: averaging over the path throws away the variance the price accumulates within the window. For a lognormal driver with full-life continuous averaging the effective variance drops by almost exactly a factor of three,

$$\sigma_{\text{eff}} \;\approx\; \frac{\sigma}{\sqrt{3}}, \tag{15}$$

(the variance of $\frac{1}{T}\int_0^T W_t\,\mathrm{d}t$ is $T/3$). Shorter windows at the end of the option's life interpolate between σ (a one-day window: the vanilla) and the full reduction, as in Figure 10. An at-the-money Asian on a 40-vol commodity prices like a 23-vol vanilla—cheap enough that corporates can hedge whole programmes with it.

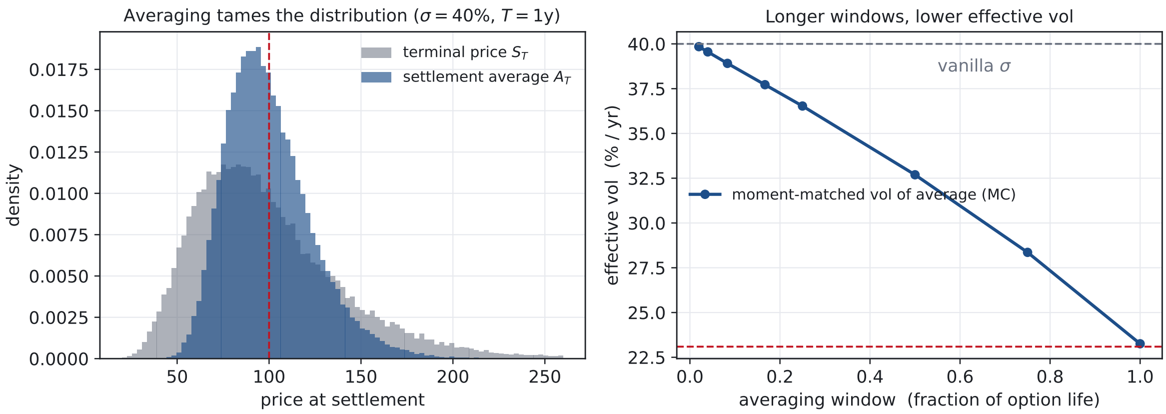

Figure 10: Monte Carlo with $\sigma = 40\%$, $T = 1\text{y}$, daily fixings, 60,000 paths. **Left:** the distribution of the full-life settlement average A_T (blue) against the terminal price S_T (grey): tighter and more symmetric. **Right:** moment-matched effective volatility of the average versus the length of the averaging window, from the vanilla σ (dashed grey) down to near $\sigma/\sqrt{3}$ (dashed red) for full-life averaging.

9.3 Pricing in practice

The arithmetic average of lognormals is not lognormal, so there is no exact Black-style formula—but it is *nearly* lognormal (Figure 10, left, shows how mild the reshaping is), and the standard desk method exploits that: compute the exact first two moments of A_T (both available in closed form for lognormal drivers, fixing by fixing), fit a lognormal to them, and apply Black-76 with the moment-matched forward and volatility. This is the Turnbull–Wakeman/Lévy approximation [10]; for the parameters commodities live at, its error is a small fraction of a vol point, inside any quoted spread. Monte Carlo handles the rest (in-progress windows, seasonally varying vol, smile), with hedge ratios obtained by re-pricing under a small shift of each futures input in turn. Appendix D includes a compact implementation of both routes.

One bookkeeping subtlety earns a warning. Once inside the averaging window, part of A_T is already *known*; the option is progressively a claim on fewer and fewer unknown fixings, its effective volatility draining away day by day. Desks carry "in-fixing" Asians as options on the remaining average with an adjusted strike, and the Greeks decay accordingly—delta migrates from the futures strip into realised cash.

10 Spread options: the correlation is the price

10.1 Margins are spreads

The commodity world's industrial logic is transformation: crude into products, gas into power, soybeans into meal and oil, one delivery point or month into another. Each transformation has a margin, and each margin is a *spread* between two commodity prices. The natural hedges and options follow:

- **Crack spread** (refining): product price minus crude price.
- **Spark spread** (generation): power price minus heat rate × gas price—the running margin of a gas turbine.
- **Calendar spread** (defined in §3.3): the same commodity, two delivery months—the difference storage converts into profit (§11).
- **Locational spread**: the same commodity, two places—the difference pipelines and freight convert into profit; the basis of §3.2 viewed as a tradable object.

An option on a spread, payoff $(F_1(T) - F_2(T) - K)^+$, is thus not an exotic curiosity: it is the option form of a refinery, a power plant, a storage cavern, a pipeline. Real-asset valuation and hedging in commodities is substantially the business of pricing spread options.

10.2 Margrabe, Kirk, and the shape of the problem

Model both futures as driftless lognormals with vols σ_1, σ_2 and correlation ρ. The spread itself is *not* lognormal—it crosses zero, which is not a defect but the point (margins go negative; that is when plants shut down)—so Black-76 on the spread is unavailable. Two classical results carry the desk.

Margrabe ($K = 0$) [8]. The option to exchange one asset for another has an exact Black-style price: measuring one asset in units of the other makes the problem one-dimensional, and

$$
\begin{aligned}
C &= e^{-r\tau}\,[\,F_1\,N(d_1) - F_2\,N(d_2)\,], \qquad d_{1,2} = \frac{\ln(F_1/F_2) \pm \frac{1}{2}\hat{\sigma}^2\tau}{\hat{\sigma}\sqrt{\tau}}, \\
\hat{\sigma}^2 &= \sigma_1^2 - 2\rho\,\sigma_1\sigma_2 + \sigma_2^2.
\end{aligned} \tag{16}
$$

Kirk ($K \neq 0$) [9]. For modest K, absorb the strike into the second asset: treat $F_2 + K$ as approximately lognormal with volatility scaled by the lognormal fraction, $\sigma_2 F_2/(F_2 + K)$, and apply (16) with $F_2 \to F_2 + K$. The approximation is excellent when K is small against F_2 (the usual case for margin strikes) and degrades as K grows; beyond it lie numerical integration and Monte Carlo, which the products of §11 need anyway.

10.3 Correlation risk

Look at the structure of $\hat{\sigma}$ in (16): with $\sigma_1 \approx \sigma_2 = \sigma$ it collapses to $\hat{\sigma} = \sigma\sqrt{2(1-\rho)}$. The spread's volatility—and with it the option value—is controlled by $1 - \rho$, and the commodity pairs in question are *highly* correlated: crude and its products, gas and power, adjacent delivery months all co-move with ρ of 0.8–0.95+. The consequence is stark and is the single most important practical fact about these products: **the spread option is priced off the small residual $1 - \rho$, so its value is brutally sensitive to a parameter that is unobservable, unstable, and not directly hedgeable** (Figure 11). Move ρ from 0.90 to 0.95 and a spread option loses a third of its value; no standard traded instrument exists whose price pins that number down the way a futures trade pins down a price exposure. In practice trading desks infer ρ from whatever spread-option prices are visible in the market, re-value their positions under deliberately pessimistic correlation assumptions, and hold back part of each deal's apparent profit as a cushion for the day the assumed correlation proves wrong. When someone on a commodity desk says "correlation is the price," this equation is what they mean.

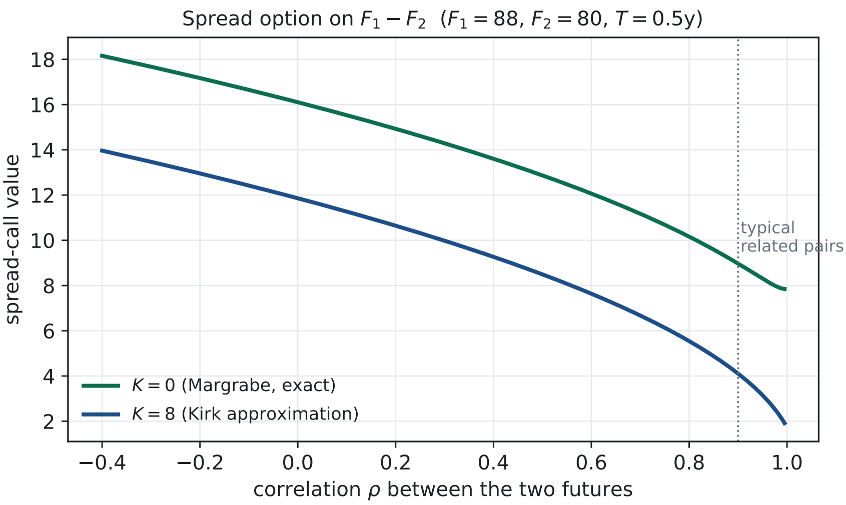

Figure 11: Spread-option value against correlation ($F_1 = 88$, $F_2 = 80$, $\sigma_1 = 38\%$, $\sigma_2 = 34\%$, $T = 0.5$y): Margrabe (16) for $K = 0$, Kirk for $K = 8$. In the region where related commodity pairs actually live (dotted line and rightward), the value falls off a cliff in ρ: the option is a position in $1 - \rho$.

A modelling footnote: a two-factor curve model prices calendar-spread options *consistently* with the vanilla surface, because (13) and the factor loadings imply the whole correlation matrix between maturities. That is the deeper reason multi-factor curve models earn their keep: they turn correlation from a free parameter into a consequence of curve dynamics—which is also why calendar-spread options are the natural calibration instruments for ρ, κ in (11).

11 Swing, take-or-pay, and storage: options on volume

The exotics considered so far modify the payoff function. The physical supply contracts introduced at the end of §3.3—swing, take-or-pay, storage leases—modify something stranger: the *quantity*. A gas supply contract does not deliver one fixed contract-per-month schedule; it gives the buyer the right to vary the volume taken day by day—to *swing*—between contractual bounds, at a fixed price. The option is on volume, and this section prices it.

11.1 Swing contracts

A stylised swing deal: over a winter, take gas at strike K on any day you choose, subject to a daily maximum $q_{\max}$, and annual minimum and maximum total volumes $[Q_{\min}, Q_{\max}]$ (the minimum is the *take-or-pay* clause of §3.3: pay for it whether or not you take it). Each day you observe the spot price and decide how much to take. Taking heavily today spends annual volume that might be worth more on a colder, dearer day next month; the constraints couple every day's decision to every other's.

This is an optimal control problem of the same family as pricing a Bermudan-style option (one exercisable on many dates): the state is (time, price factors, volume already taken), and the value function satisfies a dynamic program—on each day, choose today's volume to maximise immediate payoff plus the continuation value of the remaining volume rights. In low dimensions (one/two-factor curve models) it is solved on trees or grids; with more factors, by least-squares Monte Carlo—the Longstaff–Schwartz regression method [11]—extended with the volume state. Two structural facts give the intuition. A swing with $Q_{\min} = Q_{\max}$ is just a fixed forward purchase for every day (no optionality); one with $q_{\max}$ unconstrained collapses to a portfolio of independent daily vanilla options. Real swing lives between the two, and its value is *super-additive in freedom*: each relaxation of the constraints buys optionality, and marginal volume rights have declining value—the first flexible therm is worth the most.

11.2 Storage as a real option

A storage cavern is the physical twin of swing: the right, each day, to inject, withdraw, or wait, subject to capacity and flow-rate limits. Its value is driven by exactly the curve features this note has assembled: the *seasonal shape* (inject cheap summers, withdraw dear winters—the deterministic part of the value, lockable today with calendar spreads), *spot volatility and mean reversion* (fast reverting noise around the seasonal shape is repeatedly monetisable by a flexible asset), and *calendar-spread volatility* (the option to re-optimise as the curve moves, which only a multi-factor model prices). Valuation is the same dynamic program as swing with inventory as the volume state; the standard desk approach is either grid-based control on a one/two-factor model or Longstaff–Schwartz on simulated curves, benchmarked against the *rolling-intrinsic* lower bound (lock today's best calendar spreads, re-lock whenever the curve moves in your favour)—a simple, hedgeable strategy that in practice captures a large fraction of full option value and, importantly, tells the desk how much of the valuation is realisable without heroic assumptions.

The one-sentence summary of this section: in commodities, *physical assets are exotic options*—swing is a Bermudan on volume, storage is a strip of calendar-spread options with an inventory constraint—and the models of §6–§8 are what make their valuation and hedging coherent with the traded curve and vol surface.

12 Power: where the toolkit is rebuilt

Electricity is the limiting case that stress-tests every idea in this note, because the physical foundation of §2 is absent: power cannot be stored (pumped hydro and batteries aside, at the margins). Demand must be met by generation *in the same instant*.

12.1 Spot behaves like no other price

No storage means no inventory buffer, and the spot price is set hour by hour by the intersection of inelastic demand with a supply "stack" that turns nearly vertical at the capacity limit. The resulting behaviour (Figure 12) would be pathological anywhere else:

- **Spikes.** When demand approaches capacity, the price can jump one or two orders of magnitude for a few hours or days, then collapse back as the scarcity passes: violent upward jumps glued to violent mean reversion (κ measured in days, not months).
- **Negative prices.** Inflexible generation (nuclear, and subsidised renewables that earn per unit produced) will pay to keep producing through demand troughs; with nowhere to store the surplus, the price clears below zero.
- **Layered seasonality.** Annual, weekly and *intraday* patterns, all first-order; the traded objects are shaped accordingly (*baseload* contracts deliver power in every hour of the period, *peak* contracts only in the high-demand business hours).

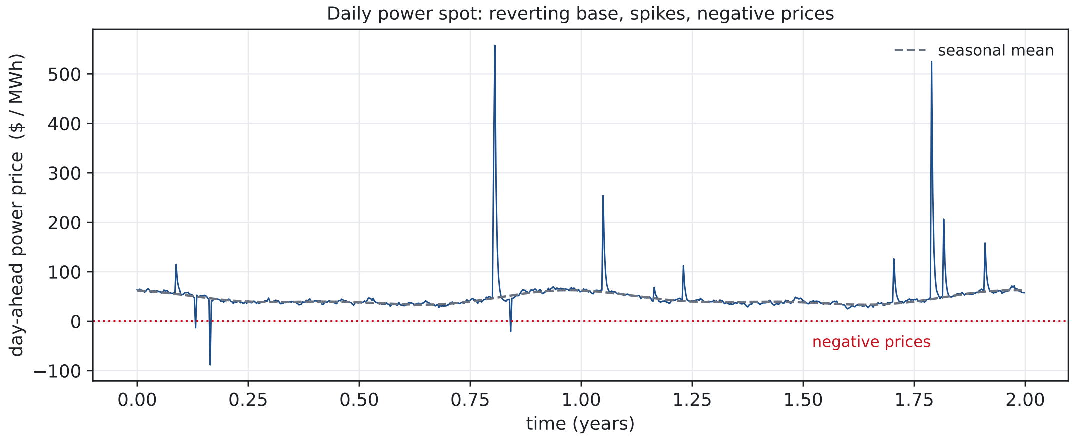

Figure 12: Two years of synthetic daily power prices: a mean-reverting base around a seasonal level (grey dashed), rare spikes an order of magnitude above it that decay within days, and occasional negative prices. No storable commodity could sustain this path: the cash-and-carry trades of §2 would flatten it.

Spot models therefore bolt jumps onto fast mean reversion (jump-diffusions in the Schwartz tradition, or regime-switching between "normal" and "spike" states), with the seasonal function carrying the calendar. Fitting them is honest work; the deeper problem is what the spot price now *means* for derivatives.

12.2 Pricing without carry

For a storable commodity, (2) welds the forward to the spot, and hedging an option means trading a claim whose price is tied to the spot. For power the weld is gone: today's spike says nothing enforceable about January's forward, and no trade converts spot power now into power later. Formally, spot power is not the price of a tradable, storable asset, so risk-neutral arguments cannot be run *through the spot* at all. The forward is not $S_t e^{(r+u-y)\tau}$ for any y; it is set directly by hedging supply and demand —generators selling forward, retailers buying—and equals expected spot only up to a risk premium that can be large (fat spike risk concentrates in whoever holds it) and sign-shifting across the year.

The escape is the move this note has been rehearsing from the start, taken to its logical end: *promote the forward curve itself to the fundamental object.* Forwards are traded, margined, storable-in-a-portfolio financial contracts, and they are martingales under the pricing measure by the argument of (3), spikes or no spikes. So:

- **Options on forwards** (the bulk of the liquid market: options on monthly, quarterly, yearly baseload/peak contracts) are priced with Black-76 and multi-factor curve models exactly as before—with the empirical caveat that delivery-period averaging (a "monthly" contract is really an average over its hours) tames the input vol, Asian-style.
- **Spot-exposed products**—caps on hourly prices, the optionality of a physical plant under a tolling agreement (§3.3: an hourly option, every hour, on power minus heat rate × gas, priced with the spread machinery of §10), contracts to supply a retail customer's whatever-it-turns-out-to-be consumption—cannot hide from the spike distribution. These are priced on spot models whose parameters are calibrated so that the model correctly reprices the forwards and options that do trade; hedged approximately, with a portfolio of forward contracts covering each delivery period; and the risk that remains—the spikes no forward position can offset—is charged for in the price and backed by a capital cushion. Hedging away the spike itself is not possible; bearing spike risk for a fee is precisely the business a power trading desk is in.

Power, in short, is where the slogan of §1 stops being a convenience and becomes the entire theory: there is only the curve.

13 The map, and where to go next

The route we have taken compresses to a few sentences. A commodity is a stored, consumed thing, so the tradable object is the futures curve, whose shape is carry—storage against convenience—where storage is possible, and pure hedging equilibrium where it is not. Around that curve sits a market of producers, consumers and processors whose hedging needs (§3) explain both the standard contracts—swaps and options settling on averages, spread products, volume-flexible supply deals—and the frictions (margin, basis, rolling) that shape how the contracts are used. Under the pricing measure the futures is driftless, so vanillas are Black-76 and all modelling effort concentrates where it belongs: on volatility structure (Samuelson, seasonality, inventory-driven skew) and on the joint dynamics of the curve (mean reversion, two factors, correlation across maturities). The products that make commodities distinctive—Asians, spreads, swing, storage, power structures—are in one-to-one correspondence with the physical world's averaging, transformation, flexibility and non-storability, and each leans on a specific piece of the curve machinery: Asians on the vol term structure, spreads on cross- and calendar-correlation, swing and storage on mean reversion and curve factors, power on the divorce of spot from forward.

For further reading: Schwartz [6] and Schwartz–Smith [7] are the foundational dynamics papers and remain the clearest; Clewlow–Strickland [13] is the practitioner's handbook for energy specifically; Geman [14] surveys the full breadth of commodity markets; Eydeland–Wolyniec [15] goes deepest on power and physical-asset valuation.

A The carry relation, carefully

We derive (2) and its inequality form, keeping the cash flows explicit. Time runs from t to T, $\tau = T - t$; the interest rate r and proportional storage cost u are constants.

Cash-and-carry (pins F from above). At t: borrow S_t, buy one unit of the commodity, sell one futures contract at $F(t,T)$ (costless). Over $[t,T]$: pay storage continuously; financing the position with its storage bill compounds the debt to $S_t e^{(r+u)\tau}$ by T. At T: deliver the unit into the futures contract, receive $F(t,T)$, repay the debt. Net certain payoff: $F(t,T) - S_t e^{(r+u)\tau}$. Ruling out riskless profit gives

$$F(t,T) \;\le\; S_t\, e^{(r+u)\tau}. \tag{1}$$

(If storage is better modelled as a lump cost with present value U per unit, the same trade gives $F \le (S_t + U)e^{r\tau}$; the proportional form keeps the algebra uniform with yields.)

Reverse cash-and-carry (would pin F from below). At t: borrow one unit of the commodity, sell it at S_t, invest the proceeds, buy one future. At T: collect $S_t e^{r\tau}$, pay $F(t,T)$, take delivery, return the unit (and pocket the storage the lender saved, if the lender rebates it). Net: $S_t e^{(r+u)\tau} - F(t,T)$, whose non-positivity would force equality in (1). For an investment asset (gold) this trade runs and equality holds. For a consumption asset it fails at the first step: holders of inventory will not lend it, because parting with physical stock forfeits exactly the operational optionality they hold it for. The bound is one-sided, and the convenience yield $y \ge 0$ is *defined* as the gap:

$$F(t,T) \;=\; S_t\, e^{(r+u-y)\tau}. \tag{2}$$

y is thus an implied quantity, read off the curve—observable wherever spot and futures both are, and interpretable as the marginal value of physical inventory. Empirically it rises steeply as inventories fall (the scarce barrel is the useful one), which is the mechanism behind backwardation in tight markets.

B Black-76: derivation and Greeks

B.1 The martingale method, and the two underlyings

The martingale (risk-neutral) method prices every European claim by the same two-step recipe. *Step one:* find the probability distribution of the underlying at expiry under the pricing measure $\mathbb{Q}$—the measure under which every *traded, costly* asset, measured in units of the money-market account $B_t = e^{rt}$, is a martingale (has no drift). *Step two:* the claim's value is the discounted $\mathbb{Q}$-expectation of its payoff,

$$V_t \;=\; e^{-r\tau}\,\mathbb{E}^{\mathbb{Q}}[\text{payoff at } T \,|\, \mathcal{F}_t]\,, \qquad \tau = T - t. \tag{17}$$

We now run this recipe twice in parallel—once for a stock (Black–Scholes), once for a commodity future (Black-76)—to expose exactly where they differ.

Step one for a stock. A stock is a costly traded asset: to hold it you part with S_t. Demanding that S_t/B_t be driftless under $\mathbb{Q}$ forces the stock's drift to be r (its dynamics become $\mathrm{d}S/S = r\,\mathrm{d}t + \sigma\,\mathrm{d}W$), so applying Itô to $\ln S_t$,

$$\ln S_T \;\sim\; \mathcal{N}\big(\ln S_t + \big(r - \tfrac{1}{2}\sigma^2\big)\tau,\; \sigma^2\tau\big)\,, \qquad \mathbb{E}^{\mathbb{Q}}[S_T] = S_t\, e^{r\tau}. \tag{18}$$

Step one for a future. A futures position, by contrast, is a *costless* instrument—nothing is paid to enter it, and daily settlement pays out $\mathrm{d}F$ as you go—so the martingale requirement applies to F directly, with no division by B and no drift correction: as established in (3), $\mathrm{d}F/F = \sigma\,\mathrm{d}W$ and

$$\ln F(T,T) \;\sim\; \mathcal{N}\big(\ln F - \tfrac{1}{2}\sigma^2\tau,\; \sigma^2\tau\big)\,, \qquad \mathbb{E}^{\mathbb{Q}}[F(T,T)] = F \equiv F(t,T). \tag{19}$$

Set the two side by side. Both underlyings are lognormal at expiry with the *same* variance $\sigma^2\tau$; the only difference is the mean. Both means say the same thing in different clothes—"the expected value at expiry is today's forward value of the asset"—but for the stock that forward value is $S_t e^{r\tau}$ (buy now, fund at r), while for the future it is simply the quoted F itself. Every difference between Black–Scholes and Black-76 flows from this one line: *the carry sits inside the stock's distribution but has already been absorbed into the quoted futures price.* Writing M for the mean,

$$M_{\mathrm{BS}} = S_t\, e^{r\tau}, \qquad\qquad M_{\mathrm{B76}} = F(t,T), \tag{20}$$

step two becomes a single computation done once for a general M.

B.2 The lognormal expectation, in full

Let $X = e^{m+sZ}$ with $Z \sim \mathcal{N}(0,1)$, so $\mathbb{E}[X] = e^{m+\frac{1}{2}s^2} = M$. The payoff expectation splits into two pieces,

$$\mathbb{E}\big[(X-K)^+\big] = \mathbb{E}[X\,\mathbf{1}_{X>K}] - K\,\mathbb{Q}(X>K). \tag{21}$$

The event $X > K$ is the event $Z > z^*$ with $z^* = (\ln K - m)/s$, so the second piece is immediate: $\mathbb{Q}(X > K) = N(-z^*)$. The first piece is one Gaussian integral, evaluated by completing the square in the exponent:

$$\begin{aligned} \mathbb{E}[X\,\mathbf{1}_{X>K}] &= \int_{z^*}^{\infty} e^{m+sz}\,\frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,\mathrm{d}z \\ &= e^{m+\frac{1}{2}s^2} \int_{z^*}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z-s)^2/2}\,\mathrm{d}z \\ &= e^{m+\frac{1}{2}s^2}\, N(s - z^*) \;=\; M\,N(s-z^*)\,, \end{aligned} \tag{22}$$

where the middle line uses $sz - \frac{1}{2}z^2 = \frac{1}{2}s^2 - \frac{1}{2}(z-s)^2$ and the last substitutes $w = z - s$. Expressing z^* in terms of M (using $m = \ln M - \frac{1}{2}s^2$) and assembling,

$$\mathbb{E}\big[(X-K)^+\big] = M\,N(d_1) - K\,N(d_2), \qquad d_{1,2} = \frac{\ln(M/K) \pm \frac{1}{2}s^2}{s}. \tag{23}$$

This is the entire mathematical content of both formulas: *the expected payoff of a call on a lognormal depends only on the mean M, the total standard deviation s, and the strike.*

B.3 Assembling the two formulas

Apply (23) with $s = \sigma\sqrt{\tau}$ and the two means of (20), then discount:

$$\begin{aligned} &\textbf{Black–Scholes:}\quad C = e^{-r\tau}\left[\,S_t e^{r\tau} N(d_1) - K\,N(d_2)\,\right] = S_t\,N(d_1) - Ke^{-r\tau}N(d_2), \\ &\qquad\qquad d_{1,2} = \frac{\ln(S_t/K) + \big(r \pm \frac{1}{2}\sigma^2\big)\tau}{\sigma\sqrt{\tau}}, \end{aligned} \tag{24}$$

the familiar formula, with the interest rate appearing *inside* $d_{1,2}$ because it lives inside the distribution's mean; and

$$\begin{aligned} &\textbf{Black-76:}\quad C = e^{-r\tau}\left[\,F\,N(d_1) - K\,N(d_2)\,\right], \\ &\qquad d_{1,2} = \frac{\ln(F/K) \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}, \end{aligned} \tag{25}$$

which is (6): no rate inside $d_{1,2}$, because the futures price carries no drift to account for. Formally, Black-76 is Black–Scholes applied to an asset whose "growth at r" is exactly cancelled—the same formula one gets for a stock paying a continuous dividend yield equal to r. The put follows from the same computation or from put–call parity for futures options,

$$C - P \;=\; e^{-r\tau}\,(F - K)\,, \tag{26}$$

itself immediate from $(x-K)^+ - (K-x)^+ = x - K$ inside the discounted expectation.

B.4 Greeks

Differentiating (6) (using $F\,\phi(d_1) = K\,\phi(d_2)$ with ϕ the normal density, the identity that collapses most of the terms):

$$\begin{aligned} \Delta &= \frac{\partial C}{\partial F} = e^{-r\tau}N(d_1) & \Gamma &= \frac{\partial^2 C}{\partial F^2} = e^{-r\tau}\frac{\phi(d_1)}{F\sigma\sqrt{\tau}} \\ \text{Vega} &= \frac{\partial C}{\partial \sigma} = e^{-r\tau}F\,\phi(d_1)\sqrt{\tau} & \rho_{\text{rate}} &= \frac{\partial C}{\partial r} = -\tau\,C \\ \Theta &= \frac{\partial C}{\partial t} = rC \;-\; e^{-r\tau}\,\frac{F\,\phi(d_1)\,\sigma}{2\sqrt{\tau}}. \end{aligned} \tag{27}$$

Three desk-relevant readings. The delta is in *futures contracts*—a costless hedge, so the option book carries no funding position against its delta, unlike equities. The rate sensitivity is just $-\tau C$: rates enter only through discounting the premium, confirming their spectator role. And theta at the money is almost pure gamma rent, $-F\sigma\phi(0)e^{-r\tau}/(2\sqrt{\tau})$, with none of the equity cost-of-carry cross-terms.

C The Schwartz models, solved

C.1 One factor

For (9), apply Itô to $e^{\kappa t} X_t$:

$$\mathrm{d}\big(e^{\kappa t} X_t\big) = \kappa\hat{\theta} e^{\kappa t}\,\mathrm{d}t + \sigma e^{\kappa t}\,\mathrm{d}W_t, \tag{28}$$

and integrate over $[t, T]$:

$$X_T \;=\; e^{-\kappa\tau} X_t + \hat{\theta}\left(1 - e^{-\kappa\tau}\right) + \sigma\int_t^T e^{-\kappa(T-s)}\,\mathrm{d}W_s, \qquad \tau = T - t. \tag{29}$$

The Itô integral is Gaussian with variance $\sigma^2\int_t^T e^{-2\kappa(T-s)}\mathrm{d}s = \sigma^2\left(1 - e^{-2\kappa\tau}\right)/2\kappa$, giving (10). The futures price is the expectation of $S_T = e^{X_T}$, i.e. lognormal mean:

$$\ln F(t,T) = e^{-\kappa\tau} X_t + \hat{\theta}\left(1 - e^{-\kappa\tau}\right) + \frac{\sigma^2}{4\kappa}\left(1 - e^{-2\kappa\tau}\right). \tag{30}$$

Differentiating (30) with X_t the only stochastic argument: by Itô, the futures' instantaneous volatility is the loading times the spot's,

$$\left.\frac{\mathrm{d}F(t,T)}{F(t,T)}\right|_{\text{diffusion}} = e^{-\kappa(T-t)}\,\sigma\,\mathrm{d}W_t, \tag{31}$$

which is the Samuelson form (7); time-averaging its square as in (8) gives the implied-vol term structure. Note from (30) the one-factor pathologies quoted in §6: all maturities are driven by the single X_t (perfect correlation), and $\ln F \to \hat{\theta} + \sigma^2/4\kappa$ as $\tau \to \infty$ (frozen long end).

C.2 Two factors (Schwartz–Smith)

The system (11) solves factor by factor: $\xi_T = \xi_t + \mu\tau + \sigma_\xi(W_T^\xi - W_t^\xi)$, and χ_T as in (29) with $\hat{\theta} = 0$. Then $\ln S_T = \chi_T + \xi_T$ is Gaussian with

$$\begin{aligned} \mathbb{E}[\ln S_T \,|\, \mathcal{F}_t] &= e^{-\kappa\tau}\chi_t + \xi_t + \mu\tau, \\ \operatorname{Var}(\ln S_T \,|\, \mathcal{F}_t) &= \sigma_\chi^2\,\frac{1 - e^{-2\kappa\tau}}{2\kappa} + \sigma_\xi^2\,\tau + 2\rho\,\sigma_\chi\sigma_\xi\,\frac{1 - e^{-\kappa\tau}}{\kappa}, \end{aligned} \tag{32}$$

and $F(t,T) = \mathbb{E}[S_T|\mathcal{F}_t]$ gives (12) with $A(\tau) = \mu\tau + \frac{1}{2}\operatorname{Var}(\ln S_T|\mathcal{F}_t)$ evaluated at the current t (all deterministic). The loadings $e^{-\kappa\tau}$ and 1 in (12) are inherited directly from the solution, and applying Itô as above yields the futures volatility (13) and, between two maturities $\tau_1 < \tau_2$, an instantaneous correlation strictly below one—the quantity that gives calendar-spread options their value. In risk-neutral use, μ (and a market-price-of-risk shift in χ's reversion level) are absorbed into fitting the observed curve, exactly as $\hat{\theta}$ was in one factor.

D A compact numpy pricer

The routines below reproduce every number in this note: Black-76 (6), Kirk (16)-with-strike, the moment-matched Asian of §9, and a Schwartz–Smith Monte Carlo for anything the formulas do not reach. (`norm.cdf` is the standard normal distribution function from `scipy.stats`.)

```
import numpy as np
from scipy.stats import norm

def black76(F, K, sigma, T, r, cp=+1):
```

```
    """European option on a future; cp=+1 call, -1 put."""
    sT = sigma * np.sqrt(T)
    d1 = (np.log(F / K) + 0.5 * sT**2) / sT
    d2 = d1 - sT
    return cp * np.exp(-r * T) * (F * norm.cdf(cp * d1)
                                  - K * norm.cdf(cp * d2))

def kirk(F1, F2, K, s1, s2, rho, T, r):
    """Spread call on F1 - F2 with strike K (K=0: Margrabe, exact)."""
    Fe = F2 + K
    se = np.sqrt(s1**2 - 2*rho*s1*s2*(F2/Fe) + (s2*F2/Fe)**2)
    d1 = (np.log(F1/Fe) + 0.5*se**2*T) / (se*np.sqrt(T))
    d2 = d1 - se*np.sqrt(T)
    return np.exp(-r*T) * (F1*norm.cdf(d1) - Fe*norm.cdf(d2))

def asian_mm(F, K, sigma, T, r, n=21):
    """Average-price call, n fixings over [0,T]; moment-matched
    lognormal (Turnbull-Wakeman / Levy) on a driftless future."""
    t = T * np.arange(1, n + 1) / n
    M1 = F                                   # E[A]
    v = np.add.outer(0*t, 0*t) + sigma**2 * np.minimum.outer(t, t)
    M2 = (F**2) * np.exp(v).mean()           # E[A^2]
    s_eff = np.sqrt(np.log(M2 / M1**2) / T)
    return black76(M1, K, s_eff, T, r)

def ss_paths(chi0, xi0, kappa, s_chi, s_xi, rho, mu, T, n, npaths,
             seed=7):
    """Schwartz-Smith log-spot paths: ln S = chi + xi."""
    rng = np.random.default_rng(seed)
    dt = T / n
    chi = np.full(npaths, chi0); xi = np.full(npaths, xi0)
    for _ in range(n):
        z1 = rng.standard_normal(npaths)
        z2 = rho*z1 + np.sqrt(1 - rho**2)*rng.standard_normal(npaths)
        chi += -kappa*chi*dt + s_chi*np.sqrt(dt)*z1
        xi  += mu*dt + s_xi*np.sqrt(dt)*z2
    return np.exp(chi + xi)
```

Sanity checks worth running once: `black76` against put–call parity; `kirk` with `K=0` against Margrabe computed directly; `asian_mm` with `n=1` against `black76` (a one-fixing average is the vanilla); and the Monte Carlo mean of `ss_paths` against the closed-form curve (12).

E Glossary

Average-price option Option whose payoff depends on the average of daily index prices over a period rather than one date's price; the commodity vanilla (§9).

Backwardation Forward curve falling with maturity; prompt scarcity, $y > r + u$.

Basis / basis risk The price difference between the thing actually held (grade, location, month) and the benchmark contract used to hedge it; the risk that this difference moves.

Black-76 The vanilla formula for European options on futures, (6).

Calendar spread Position in two delivery months of one commodity, $F(t, T_1) - F(t, T_2)$; a bet on the curve's shape, and the building block of storage value (§3.3).

Collar Bought floor financed by a sold cap (or vice versa); locks the realised price into a band at zero upfront premium.

Contango Forward curve rising with maturity; ample inventory, $y < r + u$.

Convenience yield (y) Implicit dividend of holding physical inventory rather than the futures; the gap in the carry relation (2).

Cost of carry Interest plus storage minus convenience yield; the exponent tying futures to spot for storables. At *full carry* the futures price reimburses interest and storage completely—the ceiling of (1).

Crack / spark spread Refining margin (products minus crude) / generation margin (power minus heat rate × gas).

Fungibility Interchangeability of units of a commodity; manufactured by a contract's grade and delivery-point specification (§1.2).

Heat rate Fuel needed per unit of electricity generated; the conversion ratio inside a spark spread.

Implied volatility The σ that makes Black-76 reproduce an option's market price; plotted against strike it traces the *smile*, whose asymmetry is the *skew* (§5.2).

Kirk / Margrabe Approximate / exact formulas for spread options, (16).

Prompt (front) contract The nearest-delivery futures contract.

Rolling Replacing an expiring futures position with the same position in a later month; *stack-and-roll* hedges long-dated exposure with rolled near contracts (§3.2).

Roll yield Gain or loss generated by rolling along a sloped curve: positive in backwardation, negative in contango (§2).

Samuelson effect Futures volatility rising as a contract approaches delivery, (7).

Strip One contract for each delivery period across a span of time (e.g. twelve monthly futures covering a year).

Swap (fixed-for-floating) Periodic exchange of a fixed price against the average of a published index; the OTC hedging workhorse (§3.3).

Swaption Option to enter a swap at a pre-agreed fixed price.

Swing / take-or-pay Contractual right to vary delivered volume within bounds / obligation to pay for a minimum volume regardless (§3.3, priced in §11).

Term structure of volatility Implied vol as a function of option maturity; declining for mean-reverting commodities, (8).

Tolling agreement Renting a power plant's conversion capacity—fuel in, power out, for a fee; economically a strip of spark-spread options.

References

[1] F. Black. The pricing of commodity contracts. *Journal of Financial Economics*, 3:167–179, 1976.

[2] P. A. Samuelson. Proof that properly anticipated prices fluctuate randomly. *Industrial Management Review*, 6:41–49, 1965.

[3] N. Kaldor. Speculation and economic stability. *Review of Economic Studies*, 7:1–27, 1939. (Origin of the convenience-yield concept; developed for commodity markets by H. Working, 1949.)

[4] R. Gibson, E. S. Schwartz. Stochastic convenience yield and the pricing of oil contingent claims. *Journal of Finance*, 45:959–976, 1990.

[5] J. Gabillon. The term structures of oil futures prices. Oxford Institute for Energy Studies, Working Paper M17, 1991.

[6] E. S. Schwartz. The stochastic behavior of commodity prices: implications for valuation and hedging. *Journal of Finance*, 52:923–973, 1997.

[7] E. S. Schwartz, J. E. Smith. Short-term variations and long-term dynamics in commodity prices. *Management Science*, 46:893–911, 2000.

[8] W. Margrabe. The value of an option to exchange one asset for another. *Journal of Finance*, 33:177–186, 1978.

[9] E. Kirk. Correlation in the energy markets. In *Managing Energy Price Risk*, Risk Publications, 1995.

[10] S. M. Turnbull, L. M. Wakeman. A quick algorithm for pricing European average options. *Journal of Financial and Quantitative Analysis*, 26:377–389, 1991. (The lognormal moment-matching variant is due to E. Levy, 1992.)

[11] F. A. Longstaff, E. S. Schwartz. Valuing American options by simulation: a simple least-squares approach. *Review of Financial Studies*, 14:113–147, 2001.

[12] C. L. Culp, M. H. Miller. Metallgesellschaft and the economics of synthetic storage. *Journal of Applied Corporate Finance*, 7(4):62–76, 1995.

[13] L. Clewlow, C. Strickland. *Energy Derivatives: Pricing and Risk Management*. Lacima, 2000.

[14] H. Geman. *Commodities and Commodity Derivatives: Modeling and Pricing for Agriculturals, Metals and Energy*. Wiley, 2005.

[15] A. Eydeland, K. Wolyniec. *Energy and Power Risk Management: New Developments in Modeling, Pricing, and Hedging*. Wiley, 2003.